TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 340, Number 1, November 1993

WEIGHTED NORM INEQUALITIES FOR VILENKIN-FOURIER SERIES

WO-SANG YOUNG

ABSTRACT. Let $S_n f$ be the nth partial sum of the Vilenkin-Fourier series of $f \in L^1$. For $1 < p < \infty$, we characterize all weight functions w such that if $f \in L^p(w)$, $S_n f$ converges to f in $L^p(w)$. We also determine all weight functions w such that $\{S_n\}$ is uniformly of weak type $(1, 1)$ with respect to w.

1. INTRODUCTION

The Vilenkin-Fourier series is a generalization of the Walsh-Fourier series. While the Walsh functions are characters of the countable direct product of groups of order 2, the Vilenkin system consists of characters of $G = \prod_{i=0}^{\infty} Z_{p_i}$, a direct product of cyclic groups of order p_i, where p_i is any integer ≥ 2. In this paper, there is no restriction on the orders $\{p_i\}$.

Let μ be the Haar measure on G normalized by $\mu(G) = 1$. We identify G with the unit interval $(0, 1)$ in the following manner. Let $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We associate with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto and measure-preserving.

The characters of G are the finite products of an orthonormal system $\{\phi_k\}$ of functions on G, defined by $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $x = \{x_k\} \in G$, $k = 0, 1, \ldots$. To enumerate the finite products of $\{\phi_k\}$, we write each nonnegative integer n as a finite sum, $n = \sum_{k=0}^{\infty} \alpha_k m_k$, with $0 \leq \alpha_k < p_k$, and define $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G, and they form a complete orthonormal system on G.

For $f \in L^1$, let

$$S_n f(x) = \int_G f(t) \sum_{j=0}^{n-1} \chi_j(x - t)\, d\mu(t), \qquad n = 1, 2, \ldots,$$

be the nth partial sum of the Vilenkin-Fourier series of f. In [10], we showed that for $f \in L^p$, $1 < p < \infty$,

$$\lim_{n\to\infty} \int_G |S_n f - f|^p \, d\mu = 0.$$

Received by the editors September 6, 1989 and, in revised form, September 5, 1991.
1991 *Mathematics Subject Classification.* Primary 42C10; Secondary 42A20, 42A50, 43A50.

In this paper, we identify, for $1 < p < \infty$, all weight functions w such that for any $f \in L^p(w)$, we have $f \in L^1$ and

$$\lim_{n\to\infty} \int_G |S_n f - f|^p w\, d\mu = 0.$$

(Here $L^p(w)$ denotes the space of measurable functions on G such that $\|f\|_{p,w} \equiv (\int_G |f|^p w\, d\mu)^{1/p} < \infty$.)

In order to characterize these weight functions, we use the following notation. Let $\{G_k\}$ be the sequence of subgroups of G defined by

$$G_0 = G, \qquad G_k = \prod_{i=0}^{k-1}\{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \qquad k = 1, 2, \ldots.$$

On the interval $(0, 1)$, cosets of G_k are intervals of the form $(jm_k^{-1}, (j+1)m_k^{-1})$, $j = 0, 1, \ldots, m_k - 1$. For $k = 0, 1, \ldots$, $j = 1, \ldots, p_k$, let I_{jk} be the set in G_k that corresponds to the interval $(0, jm_{k+1}^{-1})$. The collection of all translates of I_{jk} in G, for all $j = 1, \ldots, p_k$, $k = 0, 1, \ldots$, is denoted by $\mathcal{I}$. In other words, a set I belongs to $\mathcal{I}$ if

(i) for some $x \in G$, $k = 0, 1, \ldots$, $I \subset x + G_k$,
(ii) I is a union of cosets of G_{k+1}, and
(iii) if we consider $x + G_k$ as a circle, I is an interval.

We shall call the sets in $\mathcal{I}$ generalized intervals.

We say that w is a weight function on G if w is measurable and $0 \le w(x) \le \infty$. We shall exclude the trivial cases $w \equiv 0$ a.e. and $w \equiv \infty$ a.e.

Our main results are as follows:

Theorem 1.1. *Let w be a weight function on G and $1 < p < \infty$. The following statements are equivalent:*

(i) *w satisfies the $A_p(G)$ condition: there is a constant C such that for every $I \in \mathcal{I}$,*

$$(1.1) \qquad \left(\frac{1}{\mu(I)}\int_I w\, d\mu\right)\left(\frac{1}{\mu(I)}\int_I w^{-1/(p-1)}\, d\mu\right)^{p-1} \le C.$$

(ii) *There is a constant C, depending only on w and p, such that for every $f \in L^p(w)$, we have $f \in L^1$ and*

$$(1.2) \qquad \int_G |S_n f|^p w\, d\mu \le C \int_G |f|^p w\, d\mu, \qquad n = 1, 2, \ldots.$$

(iii) *For every $f \in L^p(w)$, we have $f \in L^1$ and*

$$(1.3) \qquad \lim_{n\to\infty} \int_G |S_n f - f|^p w\, d\mu = 0.$$

For the case where the orders of cyclic groups are bounded, i.e., $\sup_i p_i < \infty$, J. Gosselin [5] defined the A_p condition, $1 < p < \infty$, as the one where (1.1) holds for all I that are cosets of G_k, $k = 0, 1, \ldots$. For this case, our $A_p(G)$ condition and his A_p condition are equivalent. This can be seen as follows. Suppose w satisfies the A_p condition in [5], and $I \in \mathcal{I}$. Then, for some $x \in G$, $k = 0, 1, \ldots$, $I \subset x + G_k$, and I is a union of cosets of G_{k+1}. It follows that $\mu(I) \geq \mu(G_{k+1}) = p_k^{-1}\mu(G_k)$. Hence

$$\begin{aligned}
&\left(\frac{1}{\mu(I)}\int_I w\,d\mu\right)\left(\frac{1}{\mu(I)}\int_I w^{-1/(p-1)}\,d\mu\right)^{p-1} \\
&\quad \leq p_k^p\left(\frac{1}{\mu(G_k)}\int_{x+G_k} w\,d\mu\right)\left(\frac{1}{\mu(G_k)}\int_{x+G_k} w^{-1/(p-1)}\,d\mu\right)^{p-1} \\
&\quad \leq C, \quad \text{if } \sup_i p_i < \infty.
\end{aligned}$$

For the bounded order case, the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 is a consequence of the result in [5]. For another proof, see [8].

We also have the following theorem for functions in $L^1(w)$.

Theorem 1.2. *Let w be a weight function on G. The following statements are equivalent:*

(i) *w satisfies the $A_1(G)$ condition: there is a constant C such that for every $I \in \mathcal{I}$,*

$$\frac{1}{\mu(I)}\int_I w\,d\mu \leq C \operatorname*{ess\,inf}_I w, \tag{1.4}$$

where $\operatorname{ess\,inf}_I w = \inf\{t > 0\colon \mu\{x \in I : w(x) < t\} > 0\}$.

(ii) *There is a constant C, depending only on w, such that for every $f \in L^1(w)$, we have $f \in L^1$ and*

$$\int_{\{|S_n f| > y\}} w\,d\mu \leq Cy^{-1}\int_G |f| w\,d\mu, \qquad n = 1, 2, \ldots, y > 0. \tag{1.5}$$

Our proofs of Theorems 1.1 and 1.2 consist of adapting the methods of Hunt, Muckenhoupt, and Wheeden [6] and Cordoba and Fefferman [2] to Vilenkin-Fourier series. We recall some properties of Vilenkin-Fourier series. Let $S_n^* f = \overline{\chi}_n S_n(f\chi_n)$ be the nth modified partial sum, $n = 1, 2, \ldots$. It is shown in [8] that if $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$, then

$$S_n^* f = \sum_{k=0}^{\infty} S_{\alpha_k m_k}^* f \tag{1.6}$$

and

$$S_{\alpha_k m_k}^* f(x) = \frac{1}{\mu(G_k)}\int_{x+G_k} f(t)\phi_k^{-\alpha_k}(x-t)\left(\sum_{j=0}^{\alpha_k - 1}\phi_k^j(x-t)\right)d\mu(t). \tag{1.7}$$

(If $\alpha_k = 0$, we interpret the sum on the right as zero.) $S_{\alpha_k m_k}^* f$ can be expressed in terms of certain conjugate functions of f and $f\phi_k^{\alpha_k}$, which are defined as follows. For $f \in L^1$ and $x = \{x_k\} \in G$, let the kth conjugate function of f be

$$H_k f(x) = \frac{1}{2}\frac{1}{\mu(G_k)}\int_{(x+G_k)\cap\{x_k \neq t_k\}} f(t)\cot(\pi(x_k - t_k)/p_k)\,d\mu(t), \tag{1.8}$$

$k = 0, 1, \ldots$. Then

$$
\begin{aligned}
S^*_{\alpha_k m_k} f(x) = {} & \frac{\alpha_k}{\mu(G_k)} \int_{(x+G_k)\cap\{x_k=t_k\}} f(t)\,d\mu(t) \\
& + \frac{1}{2}\phi_k^{-\alpha_k}(x)\frac{1}{\mu(G_k)} \int_{(x+G_k)\cap\{x_k\neq t_k\}} f(t)\phi_k^{\alpha_k}(t)\,d\mu(t) \\
& - \frac{1}{2}\frac{1}{\mu(G_k)} \int_{(x+G_k)\cap\{x_k\neq t_k\}} f(t)\,d\mu(t) \\
& + i\phi_k^{-\alpha_k}(x)H_k(f\phi_k^{\alpha_k})(x) - iH_k f(x).
\end{aligned} \tag{1.9}
$$

To estimate $S_n^* f$, we have to deal simultaneously with the conjugate functions of f and $f\phi_k^{\alpha_k}$, $k = 0, 1, \ldots$. In order to do this, we make the following changes in the proofs of [6 and 2]. In the proof of the weak type $(1, 1)$ result, we modify the function $b(t)$ in the Calderón-Zygmund decomposition to satisfy both $\int_I b\,d\mu = 0$ and $\int_I b\phi_k^{\alpha_k}\,d\mu = 0$, where α_k depends on I and n. This modification of the Calderón-Zygmund decomposition was introduced in [10]. In the proof of the strong type (p, p) result, the sharp function is modified as follows: the mean value of f over I is replaced by a function $f_{I,n} = A + B\phi_k^{-\alpha_k}$ (α_k depending on I and n) that satisfies $\int_I (f - f_{I,n})\,d\mu = 0$ and $\int_I (f - f_{I,n})\phi_k^{\alpha_k}\,d\mu = 0$. These modified sharp functions will be discussed in §3.

Section 2 contains some properties of $A_p(G)$. We also show that the $A_p(G)$ condition is necessary and sufficient for the $L^p(w)$-boundedness of a Hardy-Littlewood type maximal function that is appropriate for the study of Vilenkin-Fourier series.

In §3 we introduce the modified sharp functions. We obtain estimates for the modified sharp function of $S_n^* f$ in terms of the maximal function of f. We also prove weighted norm inequalities relating the maximal function and sharp functions.

The results in §§2 and 3 will be used in the proof of (i) $\Rightarrow$ (ii) of Theorem 1.1. This implication together with the implications (ii) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iii) are proved in §4.

Theorem 1.2 is proved in §5, with the implication (i) $\Rightarrow$ (ii) based on a modified Calderón-Zygmund decomposition lemma and Theorem 1.1.

In what follows, C will denote an absolute constant which may vary from line to line.

2. The condition $A_p(G)$ and the Hardy-Littlewood maximal function

The following notation will be used throughout this paper. We decompose the collection $\mathscr{I}$ of generalized intervals into disjoint subcollections in the following manner. Let $\mathscr{I}_{-1} = \{G\}$. For $k = 0, 1, \ldots$, let $\mathscr{I}_k$ be the collection of all $I \in \mathscr{I}$ such that I is a proper subset of a coset of G_k, and is a union of cosets of G_{k+1}. The collections $\mathscr{I}_k$ are disjoint, and $\mathscr{I} = \bigcup_{k=-1}^{\infty} \mathscr{I}_k$.

For $I \in \mathscr{I}$, we define the set $3I \in \mathscr{I}$ as follows. If $I = G$, let $3I = G$. For $I \in \mathscr{I}_k$, $k = 0, 1, \ldots$, there is $x \in G$ such that $I \subset x + G_k$. If $\mu(I) \geq \mu(G_k)/3$, let $3I = x + G_k$. If $\mu(I) < \mu(G_k)/3$, consider $x + G_k$ as a circle. Then I is an interval in this circle. Define $3I \in \mathscr{I}_k$ to be the interval in this circle which contains I at its center and has measure $\mu(3I) = 3\mu(I)$. In all cases, for $I \in \mathscr{I}$, $\mu(3I) \leq 3\mu(I)$.

The properties of weights in $A_p(G)$ are analogous to the properties of those that satisfy Muckenhoupt's A_p condition. (See [1 and 4].)

If $w \in A_p(G)$, $1 < p < \infty$, then w and $w^{-1/(p-1)} \in L^1$. Hence $0 < w < \infty$ a.e. and $L^p(w) \subset L^1$. Similarly, if $w \in A_1(G)$, then $w \in L^1$ and $\operatorname{ess\,inf}_G w > 0$. Hence $0 < w < \infty$ a.e. and $L^1(w) \subset L^1$.

If $w \in A_p(G)$, $1 < p < \infty$, then Hölder's inequality implies that there is a constant C such that for any $I \in \mathscr{I}$,

$$\int_{3I} w \, d\mu \leq C \int_I w \, d\mu. \tag{2.1}$$

A basic property of $A_p(G)$ weights is the reverse Hölder inequality.

Lemma 2.1. *Let $w \in A_p(G)$, $1 < p < \infty$. Then there exist $s > 1$ and a constant C such that for any $I \in \mathscr{I}$,*

$$\left(\frac{1}{\mu(I)} \int_I w^s \, d\mu\right)^{1/s} \leq \frac{C}{\mu(I)} \int_I w \, d\mu.$$

The following two lemmas are consequences of the reverse Hölder inequality.

Lemma 2.2. *Suppose $w \in A_p(G)$, $1 < p < \infty$. Then there is $1 < q < p$ such that $w \in A_q(G)$.*

We also observe that if $w \in A_p(G)$, $1 \leq p < \infty$, then $w \in A_q(G)$, $p < q < \infty$.

Lemma 2.3. *Suppose $w \in A_p(G)$, $1 < p < \infty$. Given any $\varepsilon > 0$, there is $\delta > 0$ such that for any $I \in \mathscr{I}$, and any measurable $E \subset I$ with $\mu(E) \leq \delta \mu(I)$, we have $\int_E w \, d\mu \leq \varepsilon \int_I w \, d\mu$.*

The proofs of Lemmas 2.1–2.3 are similar to those given in [1]. (See also [4].)

We now define the Hardy-Littlewood maximal function that is appropriate for the study of Vilenkin-Fourier series. For $f \in L^1$, let

$$Mf(x) = \sup_{\substack{x \in I \\ I \in \mathscr{I}}} \frac{1}{\mu(I)} \int_I |f| \, d\mu.$$

This maximal function was first introduced by P. Simon in [9]. He also showed that the maximal operator is bounded in L^p, $1 < p \leq \infty$, and is of weak type $(1, 1)$. Using the above properties of $A_p(G)$ weights, we obtain the following analogue of Muckenhoupt's theorem [7].

Theorem 2.4. *Let w be a weight function on G. For $1 < p < \infty$, the following two statements are equivalent:*

(i) $w \in A_p(G)$.

(ii) *There is a constant C, depending only on w and p, such that for every $f \in L^p(w)$, we have $f \in L^1$ and*

$$\int_G (Mf)^p w \, d\mu \leq C \int_G |f|^p w \, d\mu.$$

The following two statements are also equivalent:

(iii) $w \in A_1(G)$.

(iv) *There is a constant* C, *depending only on* w, *such that for every* $f \in L^1(w)$, *we have* $f \in L^1$ *and*

$$\int_{\{Mf>y\}} w\,d\mu \le Cy^{-1}\int_G |f|w\,d\mu, \qquad y>0.$$

Proof. The proof of this theorem follows the same lines as those in Muckenhoupt [7] and Coifman and Fefferman [1]. (See also [4].) We shall only discuss one modification which involves the generalized intervals.

To prove that the $A_p(G)$ condition is sufficient, suppose $1 \le p < \infty$ and $w \in A_p(G)$. It is enough to prove the weak type inequality

$$\int_{\{Mf>10y\}} w\,d\mu \le Cy^{-p}\int_G |f|^p w\,d\mu,\ y>0, \qquad f\in L_p(w). \tag{2.2}$$

The strong type inequality for $1<p<\infty$ then follows from Lemma 2.2 and the Marcinkiewicz interpolation theorem.

To prove (2.2), we can assume that $y \ge \|f\|_1$. (If $y < \|f\|_1$, Hölder's inequality and the $A_p(G)$ condition imply

$$\int_G w\,d\mu \le y^{-p}\|f\|_1^p \int_G w\,d\mu \le Cy^{-p}\int_G |f|^p w\,d\mu.)$$

Apply the decomposition in [10], Lemma 2 to the function f and the value y to obtain a sequence of disjoint generalized intervals $\{I_j\}$ such that

$$y < \frac{1}{\mu(I_j)}\int_{I_j} |f|\,d\mu \le 3y, \qquad j=1,2,\dots, \tag{2.3}$$

and

$$|f(x)| \le y \quad \text{for a.e. } x \notin \bigcup_j I_j. \tag{2.4}$$

We shall show that

$$\{Mf>10y\} \subset \bigcup_j (3I_j). \tag{2.5}$$

Suppose $x \notin \bigcup_j (3I_j)$. Let $I \in \mathscr{I}$ such that $x \in I$. Then

$$\begin{aligned}\int_I |f|\,d\mu &= \int_{I\cap(\bigcup_j I_j)} |f|\,d\mu + \int_{I\cap^c(\bigcup_j I_j)} |f|\,d\mu \\ &\le \sum_j \int_{I\cap I_j} |f|\,d\mu + y\mu(I),\end{aligned}$$

by (2.4). To estimate the second last term, suppose $I \cap I_j \ne \varnothing$. From this and the fact that $I \cap {}^c(3I_j) \ne \varnothing$ (since it contains x), we have $I_j \subset 3I$. (If I_j and I belong to distinct $\mathscr{I}_k$'s, then, since they are not disjoint, one is a subset of the other. From this it follows that $I_j \subset I \subset 3I$. If I_j and I belong to the same $\mathscr{I}_k$, then they belong to the same coset of G_k, and we obtain $I_j \subset 3I$ from a geometric observation.) Hence, by (2.3),

$$\begin{aligned}\sum_j \int_{I\cap I_j} |f|\,d\mu &\le \sum_{I_j\subset 3I}\int_{I_j} |f|\,d\mu \le \sum_{I_j\subset 3I} 3y\mu(I_j) \\ &\le 3y\mu(3I) \le 9y\mu(I).\end{aligned}$$

Therefore, for all such I, $\int_I |f|\,d\mu \le 10y\mu(I)$, and we have $\{Mf > 10y\} \subset \bigcup_j (3I_j)$.

The rest of the proof of (2.2) then follows as in the above references, using (2.1), (2.3) and the $A_p(G)$ condition. This concludes the proof of Theorem 2.4.

3. The sharp functions

In this section, we introduce a sequence of sharp functions $\{f_n^\#\}$, which will be used to prove (i) $\Rightarrow$ (ii) of Theorem 1.1. Our sharp functions are modifications of that in Fefferman and Stein [3]. Before we define these modified sharp functions, we first introduce the functions $f_{I,n}$ which take the place of the mean value of f over I in the original sharp function.

Let n be a nonnegative integer expressed in the form $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \le \alpha_k < p_k$. For $I \in \mathscr{I}$ and $f \in L^1$, define $f_{I,n}$ as follows. If $I = G$, let $f_{I,n} = \int_G f\,d\mu$. For $I \in \mathscr{I}_k$, $k = 0, 1, \dots$, let $f_{I,n}$ be a function of the form $f_{I,n}(x) = a_{I,n}(f) + b_{I,n}(f)\phi_k^{-\alpha_k}(x)$, where $a_{I,n}(f)$ and $b_{I,n}(f)$ are constants, such that

$$\int_I f\,d\mu = \int_I f_{I,n}\,d\mu \tag{3.1}$$

and

$$\int_I f\phi_k^{\alpha_k}\,d\mu = \int_I f_{I,n}\phi_k^{\alpha_k}\,d\mu. \tag{3.2}$$

If I is not a coset of G_{k+1}, and if $\alpha_k \ne 0$, then $|(\mu(I))^{-1}\int_I \phi_k^{\alpha_k}\,d\mu| \ne 1$, and the choice for $a_{I,n}(f)$ and $b_{I,n}(f)$ is unique. (See [10, pp. 315–316].) If I is a coset of G_{k+1}, or if $\alpha_k = 0$, then $\phi_k^{-\alpha_k}(x)$ is a constant in I. In this case, we define $a_{I,n}(f) = (\mu(I))^{-1}\int_I f\,d\mu$, and $b_{I,n}(f) = 0$. For $n = 0, 1, \dots$, $f \in L^1$, define the nth sharp function as

$$f_n^\#(x) = \sup_{\substack{x\in I\\ I\in\mathscr{I}}} \frac{1}{\mu(I)}\int_I |f - f_{I,n}|\,d\mu.$$

The functions $f_{I,n}$ and $f_n^\#$ have the following properties.

Lemma 3.1. *There is a constant C such that for any $f \in L^1$, $I \in \mathscr{I}$, $n = 0, 1, \dots$, we have*

$$|f_{I,n}(x)| \le \frac{C}{\mu(I)}\int_I |f|\,d\mu, \qquad x \in I.$$

Proof. The proof is the same as that for (25) in [10]. See [10, pp. 315–316].

In the following, we shall set $\alpha_{-1} = 0$ for the sake of convenience.

Lemma 3.2. *There is a constant C such that for any constants A and B, we have*

$$\int_I |f - f_{I,n}|\,d\mu \le C\int_I |f - A - B\phi_k^{-\alpha_k}|\,d\mu, \tag{3.3}$$

for all $f \in L^1$, $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \le \alpha_k < p_k$, *and* $I \in \mathcal{I}_k$, $k = -1, 0, 1, \ldots$.

Proof. We observe that if I is a coset of G_k, $k = 0, 1, \ldots$, or if $I \in \mathcal{I}_k$, $k = 0, 1, \ldots$ and $\alpha_k = 0$, then $\phi_k^{-\alpha_k}(x)$ is constant in I. In these cases, we may assume $B = 0$. (3.3) holds with $C = 2$ since $f_{I,n} = (\mu(I))^{-1} \int_I f\, d\mu$.

Now, suppose $I \in \mathcal{I}_k$, $k = 0, 1, \ldots$, I is not a coset of G_{k+1}, and $\alpha_k \ne 0$. Then

$$\begin{aligned}\int_I |f - f_{I,n}|\, d\mu \le &\int_I |f - A - B\phi_k^{-\alpha_k}|\, d\mu \\ &+ \int_I |(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}|\, d\mu.\end{aligned}$$

By (3.1) and (3.2),

$$\int_I (f - A - B\phi_k^{-\alpha_k})\, d\mu = \int_I \{(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}\}\, d\mu$$

and

$$\int_I (f - A - B\phi_k^{-\alpha_k})\, d\mu = \int_I \{(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}\}\phi_k^{\alpha_k}\, d\mu.$$

Therefore, by the uniqueness of the coefficients, we have

$$(f - A - B\phi_k^{-\alpha_k})_{I,n} = (a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}.$$

It then follows from Lemma 3.1 that

$$\begin{aligned}&\int_I |(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}|\, d\mu \\ &\quad = \int_I |(f - A - B\phi_k^{-\alpha_k})_{I,n}|\, d\mu \\ &\quad \le C \int_I |f - A - B\phi_k^{-\alpha_k}|\, d\mu.\end{aligned}$$

This proves Lemma 3.2.

For $f \in L^r$, $1 < r < \infty$, let

$$M_r f(x) = \sup_{\substack{x \in I \\ I \in \mathcal{I}}} \left(\frac{1}{\mu(I)} \int_I |f|^r\, d\mu \right)^{1/r}.$$

We have the following pointwise estimate concerning the nth sharp function of the nth modified partial sum $S_n^* f$.

Theorem 3.3. *Let* $1 < r < \infty$. *There is a constant* C, *depending only on* r, *such that for any* $f \in L^r$, $n = 1, 2, \ldots$,

$$(S_n^* f)_n^{\#}(x) \le C M_r f(x), \qquad x \in G.$$

Proof. Let $x \in G$, and $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \le \alpha_k < p_k$. From Lemma 3.2, it is sufficient to show that for any $I \in \mathcal{I}_k$, $k = -1, 0, 1, \ldots$, with $x \in I$, there exist constants A and B (depending on I, n and f) such that

$$\frac{1}{\mu(I)}\int_I |S_n^* f - A - B\phi_k^{-\alpha_k}|\, d\mu \le CM_r f(x), \tag{3.4}$$

where C depends only on r.

Suppose $I \in \mathscr{I}_k$, $k = -1, 0, 1, \dots$, with $x \in I$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{3I}$. Then $S_n^* f = S_n^* f_1 + S_n^* f_2$. By Theorem 1* of [10], we have

$$\begin{aligned} \frac{1}{\mu(I)}\int_I |S_n^* f_1|\, d\mu &\le \left(\frac{1}{\mu(I)}\int_G |S_n^* f_1|^r\, d\mu\right)^{1/r} \le C\left(\frac{1}{\mu(I)}\int_G |f_1|^r\, d\mu\right)^{1/r} \\ &\le C\left(\frac{1}{\mu(3I)}\int_{3I} |f|^r\, d\mu\right)^{1/r} \le CM_r f(x). \end{aligned} \tag{3.5}$$

For $I = G$, $f_2 = 0$, and (3.4) follows with $A = B = 0$. From now on, we can assume $k \ge 0$.

To estimate $S_n^* f_2$, let c be a fixed point in I, and $y \in I$. By (1.6) we have

$$S_n^* f_2(y) - S_n^* f_2(c) = \sum_{l=0}^{\infty} \{S_{\alpha_l m_l}^* f_2(y) - S_{\alpha_l m_l}^* f_2(c)\}.$$

Expand each term on the right according to (1.7). For $l \ge k+1$, $y + G_l$ and $c + G_l$ are subsets of I. Therefore f_2 vanishes in these cosets. For $0 \le l \le k$, $y + G_l = c + G_l$. Hence

$$\begin{aligned} &S_n^* f_2(y) - S_n^* f_2(c) \\ &\quad = \sum_{l=0}^{k} \frac{1}{\mu(G_l)}\int_{y+G_l} f_2(t)\left\{\phi_l^{-\alpha_l}(y-t)\left(\sum_{j=0}^{\alpha_l - 1}\phi_l^j(y-t)\right)\right. \\ &\qquad\qquad \left. - \phi_l^{-\alpha_l}(c-t)\left(\sum_{j=0}^{\alpha_l-1}\phi_l^j(c-t)\right)\right\} d\mu(t). \end{aligned}$$

Now, for any $t \in G$, $y - t$ and $c - t$ belong to the same coset of G_l, for any $0 \le l \le k$. Since ϕ_l is constant on each coset of G_{l+1}, $l = 0, 1, \dots$, we have $\phi_l(y-t) = \phi_l(c-t)$, $0 \le l \le k-1$. Therefore all integrals on the right vanish except for the term with $l = k$. We thus have

$$S_n^* f_2(y) - S_n^* f_2(c) = S_{\alpha_k m_k}^* f_2(y) - S_{\alpha_k m_k}^* f_2(c). \tag{3.6}$$

The constants A and B are defined in terms of the conjugate functions (see (1.8)). Let $A = S_n^* f_2(c) - i\phi_k^{-\alpha_k}(c) H_k(f_2\phi_k^{\alpha_k})(c)$ and $B = iH_k(f_2\phi_k^{\alpha_k})(c)$. For $y \in I$, it follows from (3.6) that

(3.7)

$$\begin{aligned} S_n^* f_2(y) - A - B\phi_k^{-\alpha_k}(y) &= S_{\alpha_k m_k}^* f_2(y) - S_{\alpha_k m_k}^* f_2(c) \\ &\quad + i\phi_k^{-\alpha_k}(c) H_k(f_2\phi_k^{\alpha_k})(c) - i\phi_k^{-\alpha_k}(y) H_k(f_2\phi_k^{\alpha_k})(c). \end{aligned}$$

Using (1.9), we express the right side as a sum of the following terms:

$$
\begin{aligned}
E_1 &= \frac{\alpha_k}{\mu(G_k)} \int_{(y+G_k)\cap\{y_k=t_k\}} f_2(t)\,d\mu(t) \\
&\quad - \frac{\alpha_k}{\mu(G_k)} \int_{(c+G_k)\cap\{c_k=t_k\}} f_2(t)\,d\mu(t), \\
E_2 &= \frac{1}{2}\phi_k^{-\alpha_k}(y)\frac{1}{\mu(G_k)} \int_{(y+G_k)\cap\{y_k\neq t_k\}} f_2(t)\phi_k^{\alpha_k}(t)\,d\mu(t) \\
&\quad - \frac{1}{2}\frac{1}{\mu(G_k)} \int_{(y+G_k)\cap\{y_k\neq t_k\}} f_2(t)\,d\mu(t) \\
&\quad - \frac{1}{2}\phi_k^{-\alpha_k}(c)\frac{1}{\mu(G_k)} \int_{(c+G_k)\cap\{c_k\neq t_k\}} f_2(t)\phi_k^{\alpha_k}(t)\,d\mu(t) \\
&\quad + \frac{1}{2}\frac{1}{\mu(G_k)} \int_{(c+G_k)\cap\{c_k\neq t_k\}} f_2(t)\,d\mu(t), \\
E_3 &= i\phi_k^{-\alpha_k}(y)\{H_k(f_2\phi_k^{\alpha_k})(y) - H_k(f_2\phi_k^{\alpha_k})(c)\}, \\
E_4 &= -i\{H_k f_2(y) - H_k f_2(c)\}.
\end{aligned}
\tag{3.8}
$$

We shall show that $E_1 = 0$, and each of $|E_2|$, $|E_3|$ and $|E_4|$ is dominated by $CMf(x)$, and hence by $CM_r f(x)$. (3.4) will then follow from (3.5), (3.7), (3.8) and these estimates.

Since f_2 vanishes in I, and hence in $(y+G_k)\cap\{y_k=t_k\}$ and $(c+G_k)\cap\{c_k=t_k\}$, we have $E_1 = 0$.

Next, the absolute value of each of the terms in E_2 is bounded by

$$(2\mu(G_k))^{-1}\int_{x+G_k} |f|\,d\mu.$$

Therefore $|E_2| \le 2Mf(x)$.

To estimate E_3, we observe that for $y \in I$,

$$H_k(f_2\phi_k^{\alpha_k})(y) = \frac{1}{2}\frac{1}{\mu(G_k)} \int_{(x+G_k)\cap^c(3I)} f(t)\phi_k^{\alpha_k}(t)\cot(\pi(y_k - t_k)/p_k)\,d\mu(t),$$

since $f_2 = 0$ in $3I$ and $(y+G_k)\cap\{y_k\neq t_k\} \supset (x+G_k)\cap{}^c(3I)$. A similar expression holds for $H_k(f_2\phi_k^{\alpha_k})(c)$. Hence

$$
\begin{aligned}
|E_3| &= |H_k(f_2\phi_k^{\alpha_k})(y) - H_k(f_2\phi_k^{\alpha_k})(c)| \\
&\le \frac{1}{2}\frac{1}{\mu(G_k)} \int_{(x+G_k)\cap^c(3I)} |f(t)| \left|\cot\left(\frac{\pi(y_k-t_k)}{p_k}\right)\right. \\
&\qquad\qquad \left. - \cot\left(\frac{\pi(c_k-t_k)}{p_k}\right)\right| d\mu(t).
\end{aligned}
\tag{3.9}
$$

Let $3^{j+1}I = 3(3^jI)$, $j = 1, 2, \ldots$. If $3I \neq x + G_k$, write $(x+G_k)\cap{}^c(3I) = \bigcup_{j=1}^{J-1}(x+G_k)\cap(3^{j+1}I\backslash 3^jI)$, where $J = \min\{j \ge 1 : 3^jI = x+G_k\}$. Now, for $1 \le j \le J-1$ and $t \in (x+G_k)\cap(3^{j+1}I\backslash 3^jI)$,

$$
\begin{aligned}
&\left|\cot\left(\frac{\pi(y_k-t_k)}{p_k}\right) - \cot\left(\frac{\pi(c_k-t_k)}{p_k}\right)\right| \\
&\quad = \left|\sin\left(\frac{\pi(y_k-c_k)}{p_k}\right) \Big/ \left\{\sin\left(\frac{\pi(y_k-t_k)}{p_k}\right)\sin\left(\frac{\pi(c_k-t_k)}{p_k}\right)\right\}\right| \\
&\quad \le C\mu(I)\mu(G_k)/(\mu(3^{j-1}I))^2 \le C3^{-j}\mu(G_k)/\mu(3^{j+1}I).
\end{aligned}
$$

Substituting these into (3.9), we obtain

$$|E_3| \le C \sum_{j=1}^{J-1} 3^{-j} \frac{1}{\mu(3^{j+1}I)} \int_{3^{j+1}I} |f(t)|\, d\mu(t)$$
$$\le C \sum_{j=1}^{J-1} 3^{-j} Mf(x) \le CMf(x).$$

The same argument shows that $|E_4| \le CMf(x)$. This completes the proof of Theorem 3.3.

We have the following weighted norm inequality relating the maximal function and the sharp functions.

Theorem 3.4. *Let* $1 < p < \infty$, $w \in A_p(G)$. *There is a constant* C, *depending only on* p *and* w, *such that for any* $f \in L^p(w)$, $n = 0, 1, \ldots$,

$$\int_G (Mf)^p w\, d\mu \le C \int_G (f_n^{\#})^p w\, d\mu + C \left(\int_G |f|\, d\mu \right)^p \int_G w\, d\mu. \tag{3.10}$$

Proof. Let $f \in L^p(w)$. We observe that $w \in A_p(G)$ implies that $f \in L^1$. Let $y_0 = \|f\|_1$ and n be a nonnegative integer. For any $y > y_0$, apply the decomposition in [10, Lemma 2] to the function f and the value y to obtain a sequence of disjoint generalized intervals $\{I_j\}_{j \ge 1}$ such that

$$y < \frac{1}{\mu(I_j)} \int_{I_j} |f|\, d\mu \le 3y, \qquad j = 1, 2, \ldots, \tag{3.11}$$

and

$$|f(x)| \le y \quad \text{for a.e. } x \notin \bigcup_j I_j.$$

From Lemma 3.1, there is a constant $A > 1$, independent of f, such that for any $I \in \mathcal{I}$, $n = 0, 1, \ldots$,

$$|f_{I,n}(x)| \le \frac{A}{\mu(I)} \int_I |f|\, d\mu, \qquad x \in I. \tag{3.12}$$

Since $y < 4Ay$, the above decomposition applied to f and the value $4Ay$ gives a sequence of disjoint generalized intervals $\{I_{ij}\}_{i,j \ge 1}$ such that $\bigcup_i I_{ij} \subset I_j$, $j = 1, 2, \ldots$,

$$4Ay < \frac{1}{\mu(I_{ij})} \int_{I_{ij}} |f|\, d\mu \le 12Ay, \qquad i, j = 1, 2, \ldots, \tag{3.13}$$

and

$$|f(x)| \le 4Ay \quad \text{for a.e. } x \notin \bigcup_{i,j} I_{ij}.$$

Let $\delta > 0$ be a constant whose value is to be chosen later. For each I_j, there are two possibilities.

Case 1. $I_j \subset \{f_n^{\#} > \delta Ay\}$. Then

$$\sum_i \int_{I_{ij}} w\, d\mu \le \int_{\{f_n^{\#} > \delta Ay\} \cap I_j} w\, d\mu.$$

Case 2. $I_j \not\subset \{f_n^\# > \delta A y\}$. Then

$$\frac{1}{\mu(I_j)} \int_{I_j} |f - f_{I_j,n}|\, d\mu \le \delta A y .$$

Hence

$$\delta A y \mu(I_j) \ge \sum_i \left(\int_{I_{ij}} |f|\, d\mu - \int_{I_{ij}} |f_{I_j,n}|\, d\mu \right) . \tag{3.14}$$

By (3.13), we have

$$\int_{I_{ij}} |f|\, d\mu \ge 4 A y \mu(I_{ij}) .$$

Also, from (3.12) and (3.11), we have, for $x \in I_{ij} \subset I_j$,

$$|f_{I_j,n}(x)| \le A(\mu(I_j))^{-1} \int_{I_j} |f|\, d\mu \le 3Ay .$$

Hence

$$\int_{I_{ij}} |f_{I_j,n}|\, d\mu \le 3 A y \mu(I_{ij}) .$$

Substituting these estimates into (3.14), we obtain

$$\mu\left(\bigcup_i I_{ij} \right) \le \delta \mu(I_j) .$$

For $\varepsilon > 0$ to be chosen later, let δ be the value that associates with ε in Lemma 2.3. Then

$$\sum_i \int_{I_{ij}} w\, d\mu \le \varepsilon \int_{I_j} w\, d\mu .$$

Summing over all I_j in Cases 1 and 2, we obtain

$$\sum_{i,j} \int_{I_{ij}} w\, d\mu \le \int_{\{f_n^\# > \delta A y\}} w\, d\mu + \varepsilon \int_{\bigcup_j I_j} w\, d\mu .$$

Now, from (2.5), we have $\{Mf > 40Ay\} \subset \bigcup_{i,j}(3I_{ij})$. Hence, by (2.1), there is a constant B such that

$$\int_{\{Mf > 40Ay\}} w\, d\mu \le B \sum_{i,j} \int_{I_{ij}} w\, d\mu .$$

Also, $\bigcup_j I_j \subset \{Mf > y\}$. Therefore, for $y \ge y_0$, we have

$$\int_{\{Mf > 40Ay\}} w\, d\mu \le B \int_{\{f_n^\# > \delta A y\}} w\, d\mu + B\varepsilon \int_{\{Mf > y\}} w\, d\mu .$$

Multiply both sides of the above inequality by py^{p-1} and integrate over $y \in [y_0, \infty)$. We obtain

$$\begin{aligned} p \int_{y_0}^{\infty} y^{p-1} & \int_{\{Mf > 40Ay\}} w\, d\mu\, dy \\ & \le B(\delta A)^{-p} \int_G (f_n^\#)^p w\, d\mu + B\varepsilon \int_G (Mf)^p w\, d\mu . \end{aligned}$$

Also,

$$p\int_0^{y_0} y^{p-1}\int_{\{Mf>40Ay\}} w\,d\mu\,dy \le \left(\int_G |f|\,d\mu\right)^p \int_G w\,d\mu.$$

Therefore, we have

$$\begin{aligned}\int_G (Mf)^p w\,d\mu \le B(40/\delta)^p &\int_G (f_n^{\#})^p w\,d\mu \\ &+ B\varepsilon(40A)^p\int_G (Mf)^p w\,d\mu \\ &+ (40A)^p\left(\int_G |f|\,d\mu\right)^p\int_G w\,d\mu.\end{aligned}$$

Since $f \in L^p(w)$ and $w \in A_p(G)$, we have $Mf \in L^p(w)$, by Theorem 2.4. Choosing $\varepsilon = (2B)^{-1}(40A)^{-p}$, we obtain (3.10). This completes the proof of Theorem 3.4.

4. Proof of Theorem 1.1

We shall prove Theorem 1.1 by showing (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof of (i) $\Rightarrow$ (ii). Suppose $w \in A_p(G)$. We have $L^p(w) \subset L^1$. It is sufficient to prove (1.2) with S_n replaced by S_n^*. Let $f \in L^p(w)$. We observe that $S_n^* f \in L^p(w)$, since for $x \in G$,

$$|S_n^* f(x)| \le n\int_G |f|\,d\mu \le n\left(\int_G |f|^p w\,d\mu\right)^{1/p}\left(\int_G w^{-1/(p-1)}\,d\mu\right)^{1-1/p},$$

by Hölder's inequality. Therefore, using the $A_p(G)$ condition, we have

$$\int_G |S_n^* f|^p w\,d\mu \le Cn^p\int_G |f|^p w\,d\mu < \infty.$$

We now apply Theorem 3.4 to the function $S_n^* f$ to get

$$\begin{aligned}\int_G |S_n^* f|^p w\,d\mu &\le \int_G (M(S_n^* f))^p w\,d\mu \\ &\le C\int_G ((S_n^* f)_n^{\#})^p w\,d\mu + C\left(\int_G |S_n^* f|\,d\mu\right)^p\int_G w\,d\mu.\end{aligned} \tag{4.1}$$

Since $w \in A_p(G)$, Lemma 2.2 shows there is $1 < r < p$ such that $w \in A_{p/r}(G)$. Hence $|f|^r \in L^{p/r}(w) \subset L^1$, and it follows from Theorem 3.3 and Theorem 2.4 that

$$\begin{aligned}\int_G ((S_n^* f)_n^{\#})^p w\,d\mu &\le C\int_G (M_r f)^p w\,d\mu \\ &= C\int_G (M|f|^r)^{p/r} w\,d\mu \le C\int_G |f|^p w\,d\mu.\end{aligned} \tag{4.2}$$

Also, by Theorem 1* of [10],

$$\int_G |S_n^* f|\,d\mu \le \left(\int_G |S_n^* f|^r\,d\mu\right)^{1/r} \le C\left(\int_G |f|^r\,d\mu\right)^{1/r}.$$

Therefore, Hölder's inequality gives

$$\begin{aligned}(4.3)\qquad &\left(\int_G |S_n^* f\,|\,d\mu\right)^p \int_G w\,d\mu \\ &\le C\left(\int_G |f\,|^p w\,d\mu\right)\left(\int_G w^{-1/(p/r-1)}\,d\mu\right)^{p/r-1}\left(\int_G w\,d\mu\right) \\ &\le C\int_G |f\,|^p w\,d\mu,\end{aligned}$$

since $w \in A_{p/r}(G)$. Substituting (4.2) and (4.3) into (4.1), we obtain

$$\int_G |S_n^* f\,|^p w\,d\mu \le C\int_G |f\,|^p w\,d\mu.$$

This completes the proof of (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (i). Suppose w satisfies (ii). As a consequence of $L^p(w) \subset L^1$, we have $w^{-1/(p-1)} \in L^1$. (Otherwise, $w^{-1/p} \notin L^{p'}$, where $1/p + 1/p' = 1$. Hence there exists $g \in L^p$ such that $gw^{-1/p} \notin L^1$, contradicting $gw^{-1/p} \in L^p(w)$.) Therefore $\|w^{-1/(p-1)}\|_{p,w}^p = \|w^{-1/(p-1)}\|_1 < \infty$.

To prove (1.1) for $I = G$, let $f = w^{-1/(p-1)}$. Then, for $x \in G$, $S_1 f(x) = \|f\|_1 = \|w^{-1/(p-1)}\|_1 > 0$, otherwise we have $w \equiv \infty$ a.e. Hence, for every $0 < y \le \|f\|_1$, it follows from (1.2) that

$$\int_G w\,d\mu = \int_{\{S_1 f > y\}} w\,d\mu \le C y^{-p}\int_G |f\,|^p w\,d\mu.$$

Thus

$$(4.4)\qquad \int_G w\,d\mu \le C\left(\int_G f\,d\mu\right)^{-p}\int_G |f\,|^p w\,du,$$

and hence

$$\left(\int_G w\,d\mu\right)\left(\int_G w^{-1/(p-1)}\,d\mu\right)^{p-1} \le C.$$

Next, consider $I \in \mathscr{I}_k$, $k = 0, 1, \ldots$. We can assume $\int_I w^{-1/(p-1)}\,d\mu > 0$, or else there is nothing to prove. First let us assume $\mu(I) \le \mu(G_k)/2$. Take $\alpha_k = [\mu(G_k)/(2\mu(I))]$, where $[a]$ is the largest integer less than or equal to a. We have $\alpha_k \ge 1$. Let $f = w^{-1/(p-1)}\chi_I$. From (1.7), we have, for $x \in G$,

$$\begin{aligned}S_{\alpha_k m_k} f(x) &= \frac{1}{\mu(G_k)}\int_I f(t)\left(\sum_{j=0}^{\alpha_k - 1}\phi_k^j(x-t)\right)d\mu(t) \\ &= \frac{1}{\mu(G_k)}\int_I f(t)\phi_k^{(\alpha_k-1)/2}(x-t)\frac{\sin(\pi\alpha_k(x_k - t_k)/p_k)}{\sin(\pi(x_k - t_k)/p_k)}\,d\mu(t).\end{aligned}$$

Hence

$$\begin{aligned}&\phi_k^{-(\alpha_k-1)/2}(x) S_{\alpha_k m_k}(f\phi_k^{(\alpha_k-1)/2})(x) \\ &\quad = \frac{1}{\mu(G_k)}\int_I f(t)\frac{\sin(\pi\alpha_k(x_k - t_k)/p_k)}{\sin(\pi(x_k - t_k)/p_k)}\,d\mu(t).\end{aligned}$$

Since for $x, t \in I$,

$$\frac{\sin(\pi\alpha_k(x_k - t_k)/p_k)}{\sin(\pi(x_k - t_k)/p_k)} \ge \frac{2\alpha_k}{\pi} \ge \frac{\mu(G_k)}{2\pi\mu(I)},$$

we have, for $x \in I$,

$$\phi_k^{-(\alpha_k-1)/2}(x) S_{\alpha_k m_k}(f\phi_k^{(\alpha_k-1)/2})(x) \geq \frac{1}{2\pi\mu(I)} \int_I f(t)\, d\mu(t).$$

Therefore, for any $0 < y < (2\pi\mu(I))^{-1} \int_I f\, d\mu$,

$$I \subset \{\phi_k^{-(\alpha_k-1)/2} S_{\alpha_k m_k}(f\phi_k^{(\alpha_k-1)/2}) > y\}.$$

Thus, by (1.2),

$$\begin{aligned}\int_I w\, d\mu &\leq \int_{\{|S_{\alpha_k m_k}(f\phi_k^{(\alpha_k-1)/2})|>y\}} w\, d\mu \\ &\leq C y^{-p} \int_G |f\phi_k^{(\alpha_k-1)/2}|^p w\, d\mu \\ &= C y^{-p} \int_I |f|^p w\, d\mu.\end{aligned}$$

It follows that

$$\int_I w\, d\mu \leq C(2\pi\mu(I))^p \left(\int_I f\, d\mu\right)^{-p} \int_I |f|^p w\, d\mu, \tag{4.5}$$

and hence

$$\left(\int_I w\, d\mu\right) \left(\int_I w^{-1/(p-1)}\, d\mu\right)^{p-1} \leq C(\mu(I))^p.$$

We have thus proved (1.1) for $I = G$, and for $I \in \mathscr{I}_k$, $k = 0, 1, \dots$, with $\mu(I) \leq \mu(G_k)/2$. In particular, since for $k = 1, 2, \dots$, every coset of G_k is in $\mathscr{I}_{k-1}$ and $\mu(G_k) \leq \mu(G_{k-1})/2$, (1.1) holds for all cosets of G_k, $k = 0, 1, \dots$.

For the case $I \in \mathscr{I}_k$, $k = 0, 1, \dots$, $\mu(I) > \mu(G_k)/2$, assume $I \subset x + G_k$, $x \in G$. Then

$$\begin{aligned}&\left(\int_I w\, d\mu\right) \left(\int_I w^{-1/(p-1)}\, d\mu\right)^{p-1} \\ &\quad \leq \left(\int_{x+G_k} w\, d\mu\right) \left(\int_{x+G_k} w^{-1/(p-1)}\, d\mu\right)^{p-1} \\ &\quad \leq C(\mu(G_k))^p \leq C 2^p (\mu(I))^p.\end{aligned}$$

This completes the proof of (ii) $\Rightarrow$ (i).

Before we prove (ii) $\Leftrightarrow$ (iii), we introduce the following notation. For $k = 0, 1, \dots$, let $\mathscr{F}_k$ be the σ-algebra generated by the cosets of G_k. We also say that a function f belongs to $\mathscr{F}_k$, or $f \in \mathscr{F}_k$, if f is measurable with respect to $\mathscr{F}_k$.

Proof of (ii) $\Rightarrow$ (iii). Let $f \in L^p(w)$. Since (ii) $\Rightarrow$ (i), we have $w \in L^1$. Therefore, given $\varepsilon > 0$, there is $g \in \mathscr{F}_k$ for some $k = 0, 1, \dots$, such that $\|f - g\|_{p,w} < \varepsilon$. Now, for $n \geq m_k$, $S_n g = g$. Hence, for all such n,

$$\begin{aligned}\|S_n f - f\|_{p,w} &\leq \|S_n(f-g)\|_{p,w} + \|S_n g - g\|_{p,w} + \|g - f\|_{p,w} \\ &\leq C\|f-g\|_{p,w} < C\varepsilon,\end{aligned}$$

by (1.2). This proves (1.3).

Proof of (iii) $\Rightarrow$ (ii). We first show that $w \in L^1$. Let $f \in L^p(w)$ such that $f \notin \mathscr{F}_k$ for all $k = 0, 1, \ldots$. (f can be obtained as follows. If $w \notin \mathscr{F}_k$ for any $k = 0, 1, \ldots$, take $f = w^{-1/p}$. Otherwise $w \in \mathscr{F}_k$ for some $k = 0, 1, \ldots$. Since $w \not\equiv \infty$ a.e., there is a coset I of G_k such that $w(x) = c < \infty$ for $x \in I$. Let f be any function not in $\mathscr{F}_k$ for all $k = 0, 1, \ldots$, $f = 0$ outside I and $\|f\|_p < \infty$.) For this f, $\hat{f}(n) = \int_G f\overline{\chi}_n \, d\mu \neq 0$ for infinitely many n. But $|\hat{f}(n)| \, \|w\|_1^{1/p} = \|\hat{f}(n)\chi_n\|_{p,w} = \|S_{n+1}f - S_n f\|_{p,w} \to 0$ as $n \to \infty$ by (1.3). This shows $w \in L^1$.

From the proof of (ii) $\Rightarrow$ (i), we know that w satisfies $L^p(w) \subset L^1$ implies that $w^{-1/(p-1)} \in L^1$. Therefore, for any $n \geq 1$, S_n is a bounded operator from $L^p(w)$ into itself, with norm not exceeding $n\|w\|_1^{1/p}\|w^{-1/(p-1)}\|_1^{1-1/p}$. Since, by (1.3), $\sup_n \|S_n f\|_{p,w} < \infty$ for every $f \in L^p(w)$, (1.2) follows from the uniform boundedness principle. The proof of Theorem 1.1 is now complete.

5. Proof of Theorem 1.2

We shall prove the weak type $(1, 1)$ result using Theorem 1.1 and the modified Calderón-Zygmund decomposition lemma obtained in [10].

Proof of (i) $\Rightarrow$ (ii). Suppose $w \in A_1(G)$. We have $L^1(w) \subset L^1$. It is sufficient to prove (1.5) with S_n replaced by S_n^*. Let $f \in L^1(w)$. If $y < \|f\|_1$, then the $A_1(G)$ condition gives

$$\int_{\{|S_n^* f| > y\}} w \, d\mu \leq \left(\int_G w \, d\mu\right) y^{-1} \int_G |f| \, d\mu \leq C y^{-1} \int_G |f| w \, d\mu .$$

Therefore we can restrict ourselves to $y \geq \|f\|_1$.

Let $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$. From Lemma 2 of [10], we obtain a sequence of disjoint generalized intervals $\{I_j\}$ such that

$$y < \frac{1}{\mu(I_j)} \int_{I_j} |f| \, d\mu \leq 3y, \qquad j = 1, 2, \ldots, \tag{5.1}$$

and

$$|f(x)| \leq y \quad \text{for a.e. } x \notin \bigcup_j I_j \equiv \Omega. \tag{5.2}$$

Moreover, there is a decomposition of $f = g + b$, with g, $b \in L^1$, such that

$$g(x) = f(x) \quad \text{for } x \notin \Omega, \tag{5.3}$$

$$|g(x)| \leq \frac{C}{\mu(I_j)} \int_{I_j} |f| \, d\mu, \qquad x \in I_j, \ j = 1, 2, \ldots, \tag{5.4}$$

$$\int_{I_j} b \, d\mu = 0, \qquad j = 1, 2, \ldots, \tag{5.5}$$

and

$$\int_{I_j} b \phi_k^{\alpha_k} \, d\mu = 0 \quad \text{for all } I_j \in \mathscr{I}_k, \ k = 0, 1, \ldots. \tag{5.6}$$

(The constant C in (5.4) is independent of f and y.)

Since

$$\int_{\{|S_n^* f|>y\}} w\,d\mu \le \int_{\{|S_n^* g|>y/2\}} w\,d\mu + \int_{\{|S_n^* b|>y/2\}} w\,d\mu,$$

we shall show that each term on the right is bounded by $Cy^{-1}\|f\|_{1,w}$.

Because $g \in L^\infty \subset L^2(w)$ and $w \in A_1(G) \subset A_2(G)$, it follows from Theorem 1.1 that

$$\begin{aligned}\int_{\{|S_n^* g|>y/2\}} w\,d\mu &\le 4y^{-2}\int_G |S_n^* g|^2 w\,d\mu \le Cy^{-2}\int_G |g|^2 w\,d\mu \\ &= Cy^{-2}\left(\int_{c\Omega}|g|^2 w\,d\mu + \sum_j \int_{I_j}|g|^2 w\,d\mu\right).\end{aligned} \tag{5.7}$$

By (5.3) and (5.2),

$$\int_{c\Omega}|g|^2 w\,d\mu \le y\int_{c\Omega}|f|w\,d\mu.$$

Also, by (5.4), (5.1) and the $A_1(G)$ condition,

$$\int_{I_j}|g|^2 w\,d\mu \le \left(\frac{C}{\mu(I_j)}\int_{I_j}|f|\,d\mu\right)^2\left(\int_{I_j} w\,d\mu\right) \le Cy\int_{I_j}|f|w\,d\mu.$$

Substituting these estimates into (5.7), we obtain

$$\int_{\{|S_n^* g|>y/2\}} w\,d\mu \le Cy^{-1}\int_G |f|w\,d\mu.$$

To estimate $S_n^* b$, let $\Omega^* = \bigcup_j(3I_j)$. From (2.1), (5.1) and the $A_1(G)$ condition, we have

$$\begin{aligned}\int_{\Omega^*} w\,d\mu &\le C\sum_j \int_{I_j} w\,d\mu \\ &\le C\sum_j\left(\int_{I_j} w\,d\mu\right)\left(y^{-1}\frac{1}{\mu(I_j)}\int_{I_j}|f|\,d\mu\right) \\ &\le Cy^{-1}\sum_j\int_{I_j}|f|w\,d\mu \le Cy^{-1}\int_G|f|w\,d\mu.\end{aligned}$$

Hence, it is sufficient to show that

$$\int_{\{x\notin\Omega^*\,:\,|S_n^* b(x)|>y/2\}} w\,d\mu \le Cy^{-1}\int_G |f|w\,d\mu. \tag{5.8}$$

To do this, we expand $S_n^* b$ as in (1.6) and (1.9). For $x \notin \Omega^*$, it follows from (5.3), (5.5) and (5.6) that the first three terms in (1.9) vanish. (See the explanation in [10, pp. 317–318].) Therefore, for $x \notin \Omega^*$, we have

$$S_n^* b(x) = i\sum_{k=0}^{\infty}\{\phi_k^{-\alpha_k}(x)H_k(b\phi_k^{\alpha_k})(x) - H_k b(x)\}.$$

(5.8) will be proved if we can show

$$\sum_{k=0}^{\infty}\int_{c\Omega^*}|H_k(b\phi_k^{\alpha_k})|w\,d\mu \le C\int_G|f|w\,d\mu, \tag{5.9}$$

and

$$\sum_{k=0}^{\infty} \int_{{}^c\Omega^*} |H_k b| w \, d\mu \leq C \int_G |f| w \, d\mu . \tag{5.10}$$

Suppose $x \notin \Omega^*$. It follows from (1.8), (5.3), (5.5) and (5.6) that

$$H_k(b\phi_k^{\alpha_k})(x) = \frac{1}{2} \frac{1}{\mu(G_k)} \sum_{\substack{I_j \subset x+G_k \\ I_j \in \mathscr{I}_k}} \int_{I_j} b(t) \phi_k^{\alpha_k}(t) \times \left\{ \cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right) \right\} d\mu(t),$$

where $t^j = \{t_k^j\}_{k \geq 0}$ is any fixed point in I_j. (See [10, p. 318].) Let I be any coset of G_k. Fubini's theorem gives

$$\int_{I \cap {}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})(x)| w(x) \, d\mu(x) \leq \frac{1}{2} \frac{1}{\mu(I)} \sum_{\substack{I_j \subset I \\ I_j \in \mathscr{I}_k}} \int_{I_j} |b(t)| \cdot \int_{I \cap {}^c(3I_j)} \left| \cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right) \right| w(x) \, d\mu(x) \, d\mu(t).$$

From the proof of Theorem 3.3 (see the estimate of $|E_3|$ below (3.9)), we have, for any $t \in I_j$, $I_j \subset I$, $I_j \in \mathscr{I}_k$,

$$\frac{1}{\mu(I)} \int_{I \cap {}^c(3I_j)} \left| \cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right) \right| w(x) \, d\mu(x) \leq CMw(t).$$

Moreover, the last term does not exceed $Cw(t)$ for a.e. $t \in I_j$, by the $A_1(G)$ condition. Therefore

$$\int_{I \cap {}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})| w \, d\mu \leq C \sum_{\substack{I_j \subset I \\ I_j \in \mathscr{I}_k}} \int_{I_j} |b| w \, d\mu . \tag{5.11}$$

Writing $b = f - g$, using (5.4) and the $A_1(G)$ condition again, we get

$$\begin{aligned} \int_{I_j} |b| w \, d\mu &\leq \int_{I_j} |f| w \, d\mu + \left(\frac{C}{\mu(I_j)} \int_{I_j} |f| \, d\mu \right) \left(\int_{I_j} w \, d\mu \right) \\ &\leq C \int_{I_j} |f| w \, d\mu . \end{aligned}$$

Substituting this into (5.11), and summing over all cosets I of G_k, and then over all k, we obtain

$$\sum_{k=0}^{\infty} \int_{{}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})| w \, d\mu \leq C \sum_{k=0}^{\infty} \sum_{I_j \in \mathscr{I}_k} \int_{I_j} |f| w \, d\mu \leq C \int_G |f| w \, d\mu .$$

This proves (5.9).

(5.10) can be proved similarly, using (5.5) instead of (5.6). With this we have completed the proof of (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (i). Suppose w satisfies the conditions in (ii). We first prove (1.4) for $I = G$ and for $I = \mathscr{I}_k$, $k = 0, 1, \ldots$, with $\mu(I) \le \mu(G_k)/2$. Let $z > \operatorname{ess\,inf}_I w$, and $E_z = \{x \in I\colon w(x) < z\}$. Then $\mu(E_z) > 0$. Define $f = \chi_{E_z}$. We have $\|f\|_{1,w} \le z\mu(E_z) < \infty$. Applying the proofs of (4.4) and (4.5) to this f, using (1.5) instead of (1.2), we get

$$\int_I w\,d\mu \le C\mu(I)\left(\int_I f\,d\mu\right)^{-1}\int_I |f|w\,d\mu \le C\mu(I)z.$$

Since this holds for any $z > \operatorname{ess\,inf}_I w$, we have

$$\frac{1}{\mu(I)}\int_I w\,d\mu \le C\operatorname*{ess\,inf}_I w.$$

This proves (1.4) for $I = G$, and for $I \in \mathscr{I}_k$, $k = 0, 1, \ldots$, with $\mu(I) \le \mu(G_k)/2$. In particular, (1.4) holds for all cosets of G_k, $k = 0, 1, \ldots$.

For $I \in \mathscr{I}_k$, $k = 0, 1, \ldots$, with $\mu(I) > \mu(G_k)/2$, let $I \subset x + G_k$, $x \in G$. Then

$$\frac{1}{\mu(I)}\int_I w\,d\mu \le \frac{2}{\mu(G_k)}\int_{x+G_k} w\,d\mu \le 2C\operatorname*{ess\,inf}_{x+G_k} w \le 2C\operatorname*{ess\,inf}_I w.$$

This proves (ii) $\Rightarrow$ (i) and concludes the proof of Theorem 1.2.

References

1. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
2. A. Cordoba and C. Fefferman, *A weighted norm inequality for singular integrals*, Studia Math. **57** (1976), 97–101.
3. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.
4. J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
5. J. A. Gosselin, *A weighted norm inequality for Vilenkin-Fourier series*, Proc. Amer. Math. Soc. **49** (1975), 349–353.
6. R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.
7. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
8. C. W. Onneweer, *Multipliers on weighted L_p-spaces over certain totally disconnected groups*, Trans. Amer. Math. Soc. **288** (1985), 347–357.
9. P. Simon, *On a maximal function*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **21** (1978), 41–44.
10. W.-S. Young, *Mean convergence of generalized Walsh-Fourier series*, Trans. Amer. Math. Soc. **218** (1976), 311–320.

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1